# ReLU Neural Networks as Hierarchical Associative Memory via Geometric Factorization

## 1. Introduction

ReLU neural networks are often described as universal function approximators or, more loosely, as systems that “learn algorithms.” However, a more precise and empirically grounded interpretation is:

> **ReLU neural networks are hierarchical associative memory systems whose apparent intelligence arises from geometric factorization of the input space.**

In this view:

- Networks do not primarily discover symbolic algorithms
- They **factorize high-dimensional geometry into structured regions**
- Generalization emerges from **interpolation over this structured geometry**

This perspective unifies:

- Piecewise linear theory
- Observed training behavior
- Grokking phenomena
- Your key argument: **“smartness = geometric factorization”**

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## 2. ReLU Networks as Conditional Linear Systems

A ReLU layer can be rewritten as:

$$x \longrightarrow D(x)(Wx + b)$$

where $D(x)$ is a diagonal gating matrix depending on the input.

Stacking layers:

$x \longrightarrow W_L D_{L-1} W_{L-1} \ldots D_1 W_1 x$

### Interpretation:

- Each input activates a **specific chain of linear transformations**
- The network computes a **conditional linear mapping**

Without ReLU:

- $W_L \ldots W_1$) collapses to a single matrix
- Capacity is strictly limited

With ReLU:

- The system generates **many input-dependent linear maps**

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# 3. From Conditional Linear Maps to Geometric Factorization

## 3.1 Piecewise Linear Geometry

ReLU networks partition input space into regions:

- Each region corresponds to a **fixed activation pattern**
- Within each region, the network is linear

Thus:

> The network constructs a **piecewise linear tiling of the input space**

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## 3.2 Geometric Factorization (Core Claim)

Your key insight can be formalized as:

> **Training a ReLU network factorizes the input space into a set of geometrically structured regions that align with the data distribution.**

This factorization has three properties:

**(1) Compression of Complexity**

- Instead of memorizing individual points, the network:
    - Groups them into regions
    - Assigns a shared linear mapping

**(2) Alignment with Data Manifolds**

- Real data lies on lower-dimensional manifolds
- The network learns regions that:
    - Approximate these manifolds
    - Respect their structure

**(3) Reuse of Linear Structure**

- Similar inputs fall into the same or nearby regions
- They share transformations → **generalization**

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# 4. Associative Memory Reinterpreted

## 4.1 Classical View

Associative memory stores discrete mappings:

- Key → Value

## 4.2 ReLU Hierarchical Memory

ReLU networks implement:

A **hierarchically indexed associative memory over geometric regions**

- Keys = regions in input space
- Values = linear transformations
- Indexing = successive ReLU gating decisions

This is not flat memory:

- It is **factorized memory**
- Organized by geometry, not enumeration

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# 5. Why Geometric Factorization Produces Generalization

This is the central mechanism behind “intelligence”:

## 5.1 Interpolation Over Structured Regions

Because regions align with data geometry:

- New inputs fall into **meaningful regions**
- Predictions reuse learned linear mappings

Thus:

> Generalization = interpolation within geometrically factorized space

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## 5.2 Comparison to Pure Memorization

If the network only memorized:

- Each input would require its own region
- No sharing → no generalization

Instead:

- Regions **merge related inputs**
- Linear maps encode **shared structure**

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## 5.3 Depth Enables Hierarchical Factorization

Each layer:

- Refines the partition
- Reorganizes geometry

Result:

- Early layers: coarse partitioning
- Later layers: fine alignment

Thus:

> Depth = **progressive geometric factorization**

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# 6. Training Dynamics as Evidence

## 6.1 Saturation After Backpropagation

Observation:

> Evolutionary algorithms fail to improve a trained network.

Explanation:

- Backprop finds a configuration where:
  - Geometric factorization already aligns with data
- Further improvement would require:
  - A **better factorization**, not small parameter tweaks

This supports:

> The network has already optimized its **geometric decomposition**

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## 6.2 Absence of Explicit Algorithms

Observation:

> No clear discovery of algorithms (e.g., sorting)

Interpretation:

- Networks do not encode symbolic procedures
- They encode:
  - **Geometric transformations**
  - **Region-based associations**

Even when behavior appears algorithmic:

- It is implemented as **structured geometry + linear maps**

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# 7. Grokking as Geometric Re-factorization

Grokking can be reinterpreted as:

- Early phase:
    - Fine-grained regions (memorization)
- Late phase:
    - Regions reorganize into **simpler geometric structure**

Thus:

> Grokking = transition from **over-fragmented geometry** to **efficient factorization**

For large datasets:

- Factorization happens earlier
- No delayed phase transition needed

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# 8. The Role of ReLU Gating

ReLU enables geometric factorization through:

## 8.1 Region Splitting

- Each neuron introduces a hyperplane
- Space is partitioned

## 8.2 Conditional Routing

- Different regions activate different subnetworks

## 8.3 Sparse Activation

- Only relevant parts of the network are used

Together:

> ReLU implements a **geometry-aware routing system**

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# 9. Synthesis: Where the “Smarts” Come From

Your key claim can now be stated precisely:

> **The intelligence of ReLU neural networks comes from their ability to factorize high-dimensional input geometry into structured regions, enabling efficient sharing of linear mappings across similar inputs.**

Not from:

- Stored algorithms
- Symbolic reasoning
- Explicit procedural logic

But from:

- **Geometric organization of data**
- **Hierarchical partitioning**
- **Reuse of linear structure**

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# 10. Final Model

A ReLU neural network is best understood as:

> A **hierarchical geometric factorization engine** that:
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> - Decomposes input space into structured regions
> - Stores linear associations within those regions
> - Generalizes via interpolation across aligned geometry

Or more intuitively:

> It is a **data-driven coordinate system** in which simple linear rules become sufficient.

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# 11. Closing Insight

This perspective resolves the tension between:

- High performance
- Lack of explicit algorithms

Because:

> The network's "reasoning" is not procedural—it is **geometric**.

And its “memory” is not a table—it is a **factorized partition of space**.

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